Cauchy Identities for s_λ

$$\sum_\lambda s_\lambda(x_1,\dots,x_n)s_\lambda(y_1,\dots,y_n) = \prod_{i,j\in[n]} \frac{1}{1-x_iy_j}$$

Let $x^\alpha y^\beta = x_1^{\alpha_1}\cdots x_n^{\alpha_n} y_1^{\beta_1}\cdots y_n^{\beta_n}$

$[x^\alpha y^\beta](\text{LHS}) = \#\{(P,Q) \mid P,Q$ SSYT of same shape $\alpha = \text{weight}(P)$
$\beta = \text{weight}(Q)\}$

$$\text{RHS} = \prod_{i,j\in[n]} \frac{1}{1-x_iy_j} = \prod_{i,j} \sum_{a_{ij}\geq 0} (x_iy_j)^{a_{ij}}$$

$$= \sum_{\substack{A=(a_{ij}) \\ \text{non-neg integer} \\ n\times n \text{ matrix}}} \prod_i x_i^{i\text{th col sum of } A} \prod_j y_j^{j\text{th colum sum of } A}$$

Robinson–Schensted–Knuth Correspondence.

$\rightsquigarrow$ have matrix A

$\rightsquigarrow$ biword w with a_{ij} entries being $\binom{i}{j}$ arranged lexicographically

$\rightsquigarrow$ do Schensted insertion $\begin{matrix} i \longrightarrow Q \\ j \longrightarrow P \end{matrix}$

Example:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow w = \begin{matrix} 1 & 2 & 2 & 2 & 3 \\ 2 & 1 & 1 & 4 & 3 \end{matrix}$$

$$\rightsquigarrow P = \begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 4 & \\ \cline{1-2} \end{array} \quad \text{and} \quad Q = \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 2 & 3 & \\ \cline{1-2} \end{array}$$

Thm (Knuth) this construction $A \longrightarrow (P, Q)$ is a bijection with needed properties.

Thm (Knuth) If $A \xmapsto{RSK} (P,Q)$

$$A^T \xmapsto{RSK} (Q,P)$$

Thm (Knuth) If λ is the shape of P & Q

λ_1 = the length of maximum weakly increasing subsequence of w

λ_1' = the length of maximum strictly decreasing subsequence of w

weak inc: $\quad \begin{array}{l} i_{a_1} \le \cdots \le i_{a_\ell} \\ j_{a_1} \le \cdots \le j_{a_\ell} \end{array}$

strict dec: $\quad \begin{array}{l} i_{a_1} < \cdots < i_{a_\ell} \\ j_{a_1} > \cdots > j_{a_\ell} \end{array}$

SSYT $\longleftrightarrow$ Gelfond Tsetlin Pattern

Example: $n = 5$

1	1	1	1	2	2	5	5
2	2	3	3	3	3		
3	3	4	4	5	5		
5	5						

$\rightsquigarrow$

$$
\begin{array}{ccccccccc}
8 & & 6 & & 6 & & 2 & & 0 \\
& 6 & & 6 & & 4 & & 0 & \\
& & 6 & & 6 & & 2 & & \\
& & & 6 & & 2 & & & \\
& & & & 4 & & & &
\end{array}
$$